KEY

Pre-Calculus 11
Lesson 3.3 ~ Completing the Square

Some problems relating to quadratic functions are best solved with an equation written in standard form, while others are best solved with an equation written in vertex form. To be effective problem solvers, we need to be able to rewrite standard form equations in vertex form, and vice versa.

The method used to reorganize $y = ax^2 + bx + c$ into $y = a(x-p)^2 + q$ is known as **completing the square.**

Process:
Write $y = 2x^2 - 12x + 11$ in the form $y = a(x-p)^2 + q$ by completing the square.

1. Remove the coefficient of x^2 as a common factor from the first two terms.

   $y = 2(x^2 - 6x) + 11$

2. Divide the coefficient of the new x-term by 2, and then square the result. Add and subtract this number inside the brackets.

   $\left(\frac{-6}{2}\right)^2 = 9 \longrightarrow y = 2(x^2 - 6x + 9 - 9) + 11$

3. Remove the last term from the brackets and combine with the constant term.

   $y = 2(x^2 - 6x + 9) - 2 \bullet 9 + 11 \longrightarrow y = 2(x^2 - 6x + 9) - 7$

4. Factor the expression in the brackets as a complete square.

   $y = 2(x-3)(x-3) - 7 \longrightarrow y = 2(x-3)^2 - 7$

5. Check your result by expanding the result.

   $y = 2(x-3)(x-3) - 7$
   $y = 2(x^2 - 3x - 3x + 9) - 7$
   $y = 2(x^2 - 6x + 9) - 7$
   $y = 2x^2 - 12x + 18 - 7$
   $y = 2x^2 - 12x + 11$ ✓

Example #1:

Write $y = x^2 + 6x - 1$ in the form $y = a(x - p)^2 + q$ by completing the square.

$y = (x^2 + 6x + 9) - 9 - 1$

$\left(\frac{6}{2}\right)^2 = 9$

$\boxed{y = (x + 3)^2 - 10}$

check:

$y = (x + 3)(x + 3) - 10$

$y = x^2 + 3x + 3x + 9 - 10$

$y = x^2 + 6x - 1$ ✓

Example #2:

Write $y = 3x^2 - 24x + 40$ in the form $y = a(x - p)^2 + q$ by completing the square.

$\left(\frac{8}{2}\right)^2 = 16$

$y = 3(x^2 - 8x) + 40$

$y = 3(x^2 - 8x + 16 - 16) + 40$

$y = 3(x - 4)^2 - 16(3) + 40$

$\boxed{y = 3(x - 4)^2 - 8}$

check:

$y = 3(x - 4)(x - 4) - 8$

$y = 3(x^2 - 4x - 4x + 16) - 8$

$y = 3(x^2 - 8x + 16) - 8$

$y = 3x^2 - 24x + 48 - 8$

$y = 3x^2 - 24x + 40$ ✓

Example #3:
Write $y = -4x^2 - 12x + 5$ in the form $y = a(x-p)^2 + q$ by completing the square.

$y = -4(x^2 + 3x) + 5$ $\qquad \left(\frac{3}{2}\right)^2 = 2.25$

$y = -4(x^2 + 3x + 2.25 - 2.25) + 5$

$y = -4(x + 1.5)^2 - 2.25(-4) + 5$

$\boxed{y = -4(x + 1.5)^2 + 14}$

check:

$y = -4(x + 1.5)(x + 1.5) + 14$

$y = -4(x^2 + 1.5x + 1.5x + 2.25) + 14$

$y = -4(x^2 + 3x + 2.25) + 14$

$y = -4x^2 - 12x - 9 + 14$

$y = -4x^2 - 12x + 5$ ✓

Example #4: Write a Quadratic Model Function
A sporting goods store sells reusable sports water bottles for \$8. At this price their weekly sales are approximately 100 items. Research says that for every \$2 increase in price, the manager can expect the store to sell five fewer water bottles.

a) Represent this situation with a quadratic function.

Let x = # of price increases

Then $R = (8 + 2x)(100 - 5x)$

$R = 800 - 40x + 200x - 10x^2$

$\boxed{R = -10x^2 + 160x + 800}$

b) Determine the maximum revenue the manager can expect based on these estimates. What selling price will give that maximum revenue? Verify your solution.

$\left(\frac{16}{2}\right)^2 = 64$

$R = -10(\underbrace{x^2 - 16x + 64} - 64) + 800$

$R = -10(x - 8)^2 + \underbrace{640 + 800}$

$R = -10(x - 8)^2 + 1440$

Vertex (maximum): $(8, 1440)$

$\uparrow \quad \uparrow$

$x \quad R$

So 8 increases (\$16) would result in the max rev (\$1440)